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Superstring partition functions in the doubled formalism

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Abstract: Computation of superstring partition functions for the non-linear sigma model on the product of a two-torus and its dual within the scope of the doubled formalism is presented. We verify that it reproduces the partition functions of the toroidally compactified type-IIA and type-IIB theories for appropriate choices of the GSO projection.

Keywords: Superspaces, String Duality, Superstrings and Heterotic Strings, Sigma Models.

Contents

1. Introduction

Duality symmetries of various kinds have proven to be extremely important in string theory. Generically, these relate different string theories. In view of their usefulness, attempts have been made to formulate string theory in a duality-invariant manner resulting into M-theory [1, 2], F-theory [3 − 5], S-theory [6, 7]. These formulations incorporated the geometric realization of dualities, based on geometric data. More recently, with the improved understanding of nongeometric backgrounds, attempts have been made to formulate a duality-invariant string theory incorporating non-geometric data in the scheme. A formalism has been proposed, in which T-duality is made manifest by doubling the compact part of the target space [8]. This formalism has been found to be consistent with many of the known results [1 − 23]. Our calculations in the present article provide further non-trivial support for the formalism, extending to supersymmetric cases. This formulation, extending the non-linear sigma model (NLSM), is referred to as the "doubled formalism". In this formulation string theory is T-duality invariant and enhances spacetime dimensions by adding extra coordinates conjugate to winding, called the dual coordinates. The bosonic string theory has been formulated in the so-called T-fold backgrounds with geometric constraints and it has been shown that upon adding a certain topological term to the action, the corresponding quantum theory is equivalent to the quantum version of the non-linear sigma model defined on a worldsheet of arbitrary genus. A generalization of the formalism to superstring theories has also been worked out [9] . Constraint-quantization of the doubled formalism has been studied too [25, 24]. In an attempt to relate results from the new theory with the usual results in string theory, the partition function for the bosonic string on a circle has been calculated in the doubled formalism [26]. In the same vein it is important to compare the results for superstrings and with targets with more than one compact dimensions.

Here we consider an $\mathcal{N} = (1, 1)$ non-linear sigma model on a doubled torus $T^2 \times \widehat{T}^2$, where the two-tori T^2 and $\widehat{T}^2$ are dual to each other in the sense mentioned before. We compute the one-loop partition function of the two-torus. A two-torus is thought of as a direct product of circles, $T^2 \simeq S^1 \times S^1$. On each of the circles the superfields are split into ones with left and right chiralities. We write down the constraint equations for superfields on the doubled torus and find that they satisfy the appropriate chirality conditions. These constraints are interpreted as the chiral superfields which is crucial for establishing the quantum consistency of the doubled theory. A supersymmetrically extended topological term is needed for the superconformal invariance of the theory. The bosonic part of the topological term contributes with an overall sign factor to the partition function. The fermionic part of the same, on the other hand, does not contribute. This is of utmost importance for the matching of the partition function with the type-II results.

In section 2 we write down the action of the doubled $\mathcal{N} = (1, 1)$ NLSM as well as the superfields along with the constraints. The equation of motion for bosons on a torus have instanton solutions. In section 3 we present the computation of the one-loop partition function for the instanton sector for bosons [26] on a two-torus. A Poisson re-summation is required for the holomorphic factorization of the partition function. These computations for the bosons yield the sum over the internal momenta. We discuss the contributions from the bosonic and fermionic oscillators to the partition function, in section 4, in terms of the well-known modular functions. The fermionic contributions to the partition function after suitable GSO projections are found to match with the type-IIA and type-IIB results. Finally in section 5 we draw conclusions from our work.

2. The $\mathcal{N} = (1, 1)$ NLSM on a doubled torus

Let us start with the non-linear sigma model action in $N = (1, 1)$ superspace on a doubled torus, $T^2 \times \widehat{T}^2$, where T^2 and $\widehat{T}^2$ are dual to each other in the sense that the torus T^2 parametrizes the compact part of the 10 dimensional target space, while the torus $\widehat{T}^2$ parametrizes the directions conjugate to the windings. In this article we calculate the partition function solely on the compact part of the target space, that is the torus T^2 and its dual. The torus T^2 has radii $R_1$and R_2 for its two circles, while the dual torus $\widehat{T}^2$ has radii $1/R_1$ and $1/R_2$, respectively. Unhatted and hatted expressions are used to define quantities on T^2 and $\widehat{T}^2$, respectively. For example, we denote the superfields on T^2 and $\widehat{T}^2$ by Φ and $\widehat{\Phi}$, respectively. The $\mathcal{N} = (1, 1)$ non-linear sigma model action generalizing the corresponding bosonic action is

$$S = \frac{\pi}{4} \int d^2z d^2\theta [g_{\mu\nu}\epsilon^{ab} D_a\Phi^\mu D_b\Phi^\nu + \widehat{g}_{\mu\nu}\epsilon^{ab} D_a\widehat{\Phi}^\mu D_b\widehat{\Phi}^\nu]. \tag{2.1}$$

The superfields are functions of $(x, \theta, \overline{\theta})$, where x represents the spacetime coordinates and θ and $\overline{\theta}$ are the mutually conjugate Grassmannian supercoordinates. The superfields are expanded in terms of scalars, Majorana-Weyl spinors and the auxiliary fields as [27]

$$\Phi^\mu(x, \theta, \overline{\theta}) = X^\mu(x) + i\theta\psi^\mu(x) + i\overline{\theta}\widetilde{\psi}^\mu(x) + \theta\overline{\theta}F^\mu(x), \tag{2.2}$$

$$\widehat{\Phi}^\mu(x, \theta, \overline{\theta}) = \widehat{X}^\mu(x) + i\theta\widehat{\psi}^\mu(x) + i\overline{\theta}\widetilde{\widehat{\psi}}^\mu(x) + \theta\overline{\theta}\widehat{F}^\mu(x). \tag{2.3}$$

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In (2.1), the measure over the Grassmann coordinates is defined as

$$d^2\theta = d\theta d\overline{\theta}, \tag{2.4}$$

where the targetspace indices are denoted by $\mu, \nu = 1, 2$ and the worldsheet indices by $a, b = 1, 2$. The supercovariant derivatives are defined as

$$D_1 = D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \tag{2.5}$$

$$D_2 = D_{\overline{\theta}} = \frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\frac{\partial}{\partial\overline{z}}. \tag{2.6}$$

where z denotes the complex coordinate of the Euclidean worldsheet, while $\overline{z}$ denotes its complex conjugate. The metric for the tori T^2 and $\widehat{T}^2$ are

$$g_{\mu\nu} = \begin{pmatrix} {R_1}^2 & 0 \\ 0 & {R_2}^2 \end{pmatrix}, \tag{2.7}$$

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} {R_1}^{-2} & 0 \\ 0 & {R_2}^{-2} \end{pmatrix}, \tag{2.8}$$

respectively.

Using the expressions (2.1) through (2.8) the action S assumes the form

$$\begin{aligned} S = \frac{\pi}{2}\int d^2z \Big[& {R_1}^2(\partial_{\overline{z}}X^1\partial_z X_1 + \psi^1\partial_{\overline{z}}\psi_1 + \widetilde{\psi}^1\partial_z\widetilde{\psi}_1 + F^1F_1) \\ & + {R_2}^2(\partial_{\overline{z}}X^2\partial_z X_2 + \psi^2\partial_{\overline{z}}\psi_2 + \widetilde{\psi}^2\partial_z\widetilde{\psi}_2 + F^2F_2) \\ & + {R_1}^{-2}(\partial_{\overline{z}}\widehat{X}^1\partial_z\widehat{X}_1 + \widehat{\psi}^1\partial_{\overline{z}}\widehat{\psi}_1 + \widetilde{\widehat{\psi}}^1\partial_z\widetilde{\widehat{\psi}}_1 + \widehat{F}^1\widehat{F}_1) \\ & + {R_2}^{-2}(\partial_{\overline{z}}\widehat{X}^2\partial_z\widehat{X}_2 + \widehat{\psi}^2\partial_{\overline{z}}\widehat{\psi}_2 + \widetilde{\widehat{\psi}}^2\partial_z\widetilde{\widehat{\psi}}_2 + \widehat{F}^2\widehat{F}_2)\Big]. \end{aligned} \tag{2.9}$$

This action is not invariant under the T-duality transformations:

$$R_i \to \frac{1}{R_i}, \quad i = 1, 2. \tag{2.10}$$

To make it T-duality invariant we introduce the geometric constraint equations as in [26]. Towards this we introduce the superfields

$$\mathbf{P}^1(x, \theta, \overline{\theta}) = R_1\Phi^1 + {R_1}^{-1}\widehat{\Phi}^1, \tag{2.11}$$

$$\mathbf{P}^2(x, \theta, \overline{\theta}) = R_2\Phi^2 + {R_2}^{-1}\widehat{\Phi}^2, \tag{2.12}$$

$$\mathbf{Q}^1(x, \theta, \overline{\theta}) = R_1\Phi^1 - {R_1}^{-1}\widehat{\Phi}^1, \tag{2.13}$$

$$\mathbf{Q}^2(x, \theta, \overline{\theta}) = R_2\Phi^2 - {R_2}^{-1}\widehat{\Phi}^2. \tag{2.14}$$

$\mathbf{P}$ and $\mathbf{Q}$ can be expanded in terms of components, such as scalars P^μ, Q^μ and Majorana-Weyl spinors ψ_p's and ψ_q's.

$$\mathbf{P}^\mu = P^\mu + i\theta{\psi_p}^\mu + i\overline{\theta}\widetilde{\psi_p}^\mu + \theta\overline{\theta}{F_p}^\mu, \tag{2.15}$$

$$\mathbf{Q}^\mu = Q^\mu + i\theta{\psi_q}^\mu + i\overline{\theta}\widetilde{\psi_q}^\mu + \theta\overline{\theta}{F_q}^\mu, \tag{2.16}$$

where $\mu, \nu = 1, 2$, and

$$P^1 = R_1 X^1 + R_1^{-1} \widehat{X}^1, \quad {\psi_p}^1 = R_1 \psi^1 + R_1^{-1} \widehat{\psi}^1, \quad \widetilde{\psi_p}^1 = R_1 \psi^1 + R_1^{-1} \widehat{\psi}^1, \tag{2.17}$$

$$Q^1 = R_1 X^1 - R_1^{-1} \widehat{X}^1, \quad {\psi_q}^1 = R_1 \psi^1 - R_1^{-1} \widehat{\psi}^1, \quad \widetilde{\psi}_q^1 = R_1 \widetilde{\psi}^1 - R_1^{-1} \widetilde{\widehat{\psi}}^1, \tag{2.18}$$

where $i = 1, 2$. A topological term containing the superfields is also added to the action (2.1) to ensure invariance under large gauge transformations corresponding to the holomorphic and anti-holomorphic U(1) currents on $T^2 \simeq S^1 \times S^1$, generalizing the topological term for the bosonic case [26]:

$$S_{\text{top}} = \pi \int d^2 z d^2 \theta [D_{\overline{\theta}} \Phi^1 D_\theta \widehat{\Phi}^1 + D_\theta \Phi^1 D_{\overline{\theta}} \widehat{\Phi}^1 + D_{\overline{\theta}} \Phi^2 D_\theta \widehat{\Phi}^2 + D_\theta \Phi^2 D_{\overline{\theta}} \widehat{\Phi}^2]. \tag{2.19}$$

The superfields $\mathbf{P}$ and $\mathbf{Q}$ are subject to the chirality constraints

$$\int d^2 \theta D_{\overline{\theta}} \mathbf{P}^\mu = 0, \tag{2.20}$$

$$\int d^2 \theta D_\theta \mathbf{Q}^\mu = 0, \tag{2.21}$$

thus making $\mathbf{P}$ and $\mathbf{Q}$ holomorphic and anti-holomorphic, respectively. Using the expressions (2.17) and (2.18), the action (2.9) becomes

$$\begin{aligned} S = \frac{\pi}{4} \int d^2 z [& (\partial_{\overline{z}} P^1 \partial_z P^1 + {\psi_p}^1 \partial_{\overline{z}} {\psi_p}^1 + \widetilde{\psi_p}^1 \partial_z \widetilde{\psi_p}^1) \\ & + (\partial_{\overline{z}} P^2 \partial_z P^2 + {\psi_p}^2 \partial_{\overline{z}} {\psi_p}^2 + \widetilde{\psi_p}^2 \partial_z \widetilde{\psi_p}^2) \\ & + (\partial_{\overline{z}} Q^1 \partial_z Q^1 + {\psi_q}^1 \partial_{\overline{z}} {\psi_q}^1 + \widetilde{\psi_q}^1 \partial_z \widetilde{\psi_q}^1) \\ & + (\partial_{\overline{z}} Q^2 \partial_z Q^2 + {\psi_q}^2 \partial_{\overline{z}} {\psi_q}^2 + \widetilde{\psi_q}^2 \partial_z \widetilde{\psi_q}^2)]. \end{aligned} \tag{2.22}$$

This action has no explicit dependence on the radii of the two two-tori and is thus manifestly T-duality invariant. The action (2.22) is quantum equivalent to that of superstring theory only if the one-loop partition functions of both theories match. This necessitated the addition of a topological term in the bosonic case. We extend the topological term by incorporating the corresponding fermionic contributions so as to preserve $\mathcal{N} = (1, 1)$ superconformal symmetry. The extended topological term takes the form

$$\begin{aligned} S_{\text{top}} = & \pi \int d^2 z \Big[-\frac{1}{2} (\partial_{\overline{z}} P^1 \partial_z Q^1 - \partial_z P^1 \partial_{\overline{z}} Q^1 + \partial_{\overline{z}} P^2 \partial_z Q^2 - \partial_z P^2 \partial_{\overline{z}} Q^2) \\ & + \frac{1}{2} (\psi_p^1 \partial_{\overline{z}} \psi_q^1 - \psi_q^1 \partial_{\overline{z}} \psi_p^1 + \psi_p^2 \partial_{\overline{z}} \psi_q^2 - \psi_q^2 \partial_{\overline{z}} \psi_p^2) \\ & + \frac{1}{2} (\widetilde{\psi}_p^1 \partial_z \widetilde{\psi}_q^1 - \widetilde{\psi}_q^1 \partial_z \widetilde{\psi}_p^1 + \widetilde{\psi}_p^2 \partial_z \widetilde{\psi}_q^2 - \widetilde{\psi}_q^2 \partial_z \widetilde{\psi}_p^2) \Big]. \end{aligned} \tag{2.23}$$

The equations of motion for the scalars obtained from the action (2.9) have instanton solutions. These solutions are classical and come from periodicity conditions on the compact bosons. They contribute to the partition function with the sums over the bosonic momenta, to which we now turn.

3. The instanton contributions

In this section we consider the contribution of the bosonic instanton sector to the partition function. To calculate the partition function for chiral bosons one needs to employ the holomorphic factorization technique so as to retain the contribution with the right holomorphic dependence. We apply here the same technique for P and Q as in [26].

In calculating the partition function, the superfields Φ in the action (2.1) are replaced by the combinations $\mathbf{L}$ and $\widehat{\mathbf{L}}$, periodic under the shifts in the momenta of the scalars along the circles of T^2. These are given by

$$\mathbf{L}^\mu = \int D_\theta \Phi^\mu dz d\theta + \int D_{\overline{\theta}} \Phi^\mu d\overline{z} d\overline{\theta} + N\alpha^\mu + M\beta^\mu + N'\alpha^\mu + M'\beta^\mu, \tag{3.1}$$

$$\widehat{\mathbf{L}}^\mu = \int D_\theta \widehat{\Phi}^\mu dz d\theta + \int D_{\overline{\theta}} \widehat{\Phi}^\mu d\overline{z} d\overline{\theta} + \widehat{N}\alpha^\mu + \widehat{M}\beta^\mu + \widehat{N}'\alpha^\mu + \widehat{M}'\beta^\mu, \tag{3.2}$$

where α and β designate the 1-cycles of the tori, corresponding to $T^2 \simeq S^1 \times S^1$. The bosonic parts of these combinations are

$$L^\mu{}_b = \int dX^\mu + N\alpha^\mu + M\beta^\mu, \tag{3.3}$$

$$\widehat{L}^\mu_b = \int d\widehat{X}^\mu + \widehat{N}\alpha^\mu + \widehat{M}\beta^\mu, \tag{3.4}$$

where

$$N = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}, \quad \widehat{N} = \begin{pmatrix} \widehat{n}_1 & 0 \\ 0 & \widehat{n}_2 \end{pmatrix}, \tag{3.5}$$

and

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \widehat{M} = \begin{pmatrix} \widehat{m}_1 & 0 \\ 0 & \widehat{m}_2 \end{pmatrix}. \tag{3.6}$$

In equations (3.3) and (3.4)

$$dX = \partial_z X dz + \partial_{\overline{z}} X d\overline{z}, \quad d\widehat{X} = \partial_z \widehat{X} dz + \partial_{\overline{z}} \widehat{X} d\overline{z} \tag{3.7}$$

The fermionic parts of $\mathbf{L}$ and $\widehat{\mathbf{L}}$ are given by

$$L^\mu_f = i\int dz d\theta [\psi^\mu_p + \theta\overline{\theta}\partial_z \widetilde{\psi}^\mu_p] + i\int d\overline{z} d\overline{\theta} [\widetilde{\psi}^\mu_p + \overline{\theta}\theta \partial_{\overline{z}} \psi^\mu_p], \tag{3.8}$$

$$\widehat{L}^\mu_f = i\int dz d\theta [\widehat{\psi}^\mu_p + \theta\overline{\theta}\partial_z \widetilde{\widehat{\psi}}^\mu_p] + i\int d\overline{z} d\overline{\theta} [\widetilde{\widehat{\psi}}^\mu_p + \overline{\theta}\theta \partial_{\overline{z}} \widehat{\psi}^\mu_p]. \tag{3.9}$$

Similarly, the superfields $\mathbf{P}$ and $\mathbf{Q}$ are also to be combined into periodic combinations with bosonic and fermionic parts as

$$\Psi^\mu_b = \int dP^\mu + (RN + R^{-1}\widehat{N})\alpha^\mu + (RM + R^{-1}\widehat{M})\beta^\mu, \tag{3.10}$$

$$\Upsilon^\mu_b = \int dQ^\mu + (RN - R^{-1}\widehat{N})\alpha^\mu + (RM - R^{-1}\widehat{M})\beta^\mu. \tag{3.11}$$

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and the fermions ψ_p and ψ_q in the same equations are replaced by

$$\Psi^\mu_f = i\int dzd\theta[\psi^\mu_p + \theta\overline{\theta}\partial z\widetilde{\psi}^\mu_p] + i\int d\overline{z}d\overline{\theta}[\widetilde{\psi}^\mu_p + \overline{\theta}\theta\partial_{\overline{z}}\psi^\mu_p], \tag{3.12}$$

$$\Upsilon^\mu_f = i\int dzd\theta[\psi^\mu_q + \theta\overline{\theta}\partial z\widetilde{\psi}^\mu_q] + i\int d\overline{z}d\overline{\theta}[\widetilde{\psi}^\mu_q + \overline{\theta}\theta\partial_{\overline{z}}\psi^\mu_q]. \tag{3.13}$$

Here

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \tag{3.14}$$

$$R^{-1} = \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{pmatrix}. \tag{3.15}$$

Rewriting the action (2.22) in terms of the periodic combinations, Ψ^μ_b's and Υ^μ_b's, and using equations (3.8)—(3.11), one can extract the terms independent of Ψ^μ_b's and Υ^μ_b's. These terms contribute to the "instanton" sum. The instanton sector of the partition function contains sum over all field configurations,

$$\begin{aligned} Z^{\rm inst}_b = \sum_{\substack{n_1,m_1,\\ \widehat{n}_1,\widehat{m}_1}} \sum_{\substack{n_2,m_2,\\ \widehat{n}_2,\widehat{m}_2}} \exp\Bigg(& -(R_1n_1 + R_1^{-1}\widehat{n}_1)^2\frac{\pi|\tau^1|^2}{4\tau^1_2} \\ & + (R_1n_1 + R_1^{-1}\widehat{n}_1)(R_1m_1 + R_1^{-1}\widehat{m}_1)\frac{\pi\tau^1_1}{2\tau^1_2} \\ & - (R_1m_1 + R_1^{-1}\widehat{m}_1)^2\frac{\pi}{4\tau^1_2} - (R_2n_2 + R_2^{-1}\widehat{n}_2)^2\frac{\pi|\tau^2|^2}{4\tau^2_2} \\ & + (R_2n_2 + R_2^{-1}\widehat{n}_2)(R_2m_2 + R_2^{-1}\widehat{m}_2)\frac{\pi\tau^2_1}{2\tau^2_2} - (R_2m_2 + R_2^{-1}\widehat{m}_2)^2\frac{\pi}{4\tau^2_2} \\ & - (R_1n_1 - R_1^{-1}\widehat{n}_1)^2\frac{\pi|\tau^1|^2}{4\tau^1_2} + (R_1n_1 - R_1^{-1}\widehat{n}_1)(R_1m_1 - R_1^{-1}\widehat{m}_1)\frac{\pi\tau^1_1}{2\tau^1_2} \\ & - (R_1m_1 - R_1^{-1}\widehat{m}_1)^2\frac{\pi}{4\tau^1_2} - (R_2n_2 - R_2^{-1}\widehat{n}_2)^2\frac{\pi|\tau^2|^2}{4\tau^2_2} \\ & + (R_2n_2 - R_2^{-1}\widehat{n}_2)(R_2m_2 - R_2^{-1}\widehat{m}_2)\frac{\pi\tau^2_1}{2\tau^2_2} - (R_2m_2 - R_2^{-1}\widehat{m}_2)^2\frac{\pi}{4\tau^2_2}\Bigg). \end{aligned} \tag{3.16}$$

Here τ^1_1 and τ^1_2 are respectively the real and complex parts of the modular parameter of the torus T^2 while τ^2_1 and τ^2_2 are their counterparts for the dual torus $\widehat{T}^2$. The fermionic parts of the periodic combinations, Ψ^μ_f and Υ^μ_f do not couple with the momenta and windings and hence do not contribute to the partition function. They simply reproduce the classical action (2.9) after being squared. The bosons in the topological term contribute only a sign

$$Z^{\rm top}_b = \prod_i \exp[i\pi(n_i\widehat{m}_i - m_i\widehat{n}_i)] \tag{3.17}$$

to the partition function, as mentioned above.

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Holomorphic factorization of the partition function calls for Poisson re-summation. For that one first has to separate the contributions from the scalars P's and Q's in terms of independent variables. Let us consider the case where the radii of the tori are $R_i^2 = \frac{\rho_i}{\lambda_i}$ with coprime integers ρ_i and λ_i and let us define $\xi_i = \rho_i \lambda_i$, $i = 1, 2$. Then [26]

$$R_i n_i \pm R_i^{-1} \widehat{n}_i = \sqrt{\xi_i} (\frac{n_i}{\lambda_i} \pm \frac{\widehat{n}_i}{\rho_i}). \tag{3.18}$$

Now we make a substitution

$$\begin{aligned} n_i &= c_i \lambda_i + \lambda_i \sigma_{\lambda_i} \\ \widehat{n}_i &= \widehat{c}_i \rho_i + \rho_i \sigma_{\rho_i}, \end{aligned} \tag{3.19}$$

where

$$c_i, \widehat{c}_i \in \mathbb{Z} \text{ and } \sigma_{\lambda_i} \in \{0, 1/\lambda_i, \cdots, \lambda_i - 1/\lambda_i\}. \tag{3.20}$$

We can write

$$\sqrt{\xi_i} (\frac{n_i}{\lambda_i} \pm \frac{\widehat{n}_i}{\rho_i}) = \sqrt{\xi_i} (c_i \pm \widehat{c}_i + \sigma_{\lambda_i} \pm \sigma_{\rho_i}). \tag{3.21}$$

A further substitution with $h_i = c_i + \widehat{c}_i$ and $l_i = c_i - \widehat{c}_i$ allows us to rewrite the sum over n_i and $\widehat{n}_i$ as sum over h_i and $l_i \in \mathbb{Z}$. Since $h_i - l_i = 2\widehat{c}_i$, we restrict to even values of $h_i - l_i$. This is done by inserting a factor of

$$\frac{1}{2} \sum_{\phi \in \{0, 1/2\}} \exp\left[2\pi i \phi (h_i - l_i)\right]$$

in (3.16). One can repeat the process for the m_i and $\widehat{m}_i$ sums and including the contribution from the bosonic parts of the topological terms, the instanton piece of the partition function becomes

$$\begin{aligned} Z_b^{\text{inst}} = \prod_i \sum_{\substack{\phi, \chi, \sigma_{\lambda_i}, \sigma_{\rho_i}, \sigma'_{\lambda_i}, \sigma'_{\rho_i} \\ h_i, l_i, s_i, t_i}} \frac{1}{16} \exp \Bigg[&- \frac{\pi \xi_i}{4} ((h_i + \sigma_{\lambda_i} + \sigma_{\rho_i})^2 \frac{|\tau^i|^2}{\tau_2^i} \\ &- 2(h_i + \sigma_{\lambda_i} + \sigma_{\rho_i})(s_i + \sigma'_{\lambda_i} + \sigma'_{\rho_i}) \frac{\tau_1^i}{\tau_2^i} + (s_i + \sigma'_{\lambda_i} + \sigma'_{\rho_i})^2 \frac{1}{\tau_2^i} \\ &+ (l_i + \sigma_{\lambda_i} - \sigma_{\rho_i})^2 \frac{|\tau^i|^2}{\tau_2^i} - 2(l_i + \sigma_{\lambda_i} - \sigma_{\rho_i})(t_i + \sigma'_{\lambda_i} - \sigma'_{\rho_i}) \frac{\tau_1^i}{\tau_2^i} \\ &+ (t_i + \sigma'_{\lambda_i} - \sigma'_{\rho_i})^2 \frac{1}{\tau_2^i}) + 2\pi i (\phi (h_i - l_i) + \chi (s_i - t_i)) \\ &+ \frac{i\pi\xi}{2} ((l_i + \sigma_{\lambda_i} - \sigma_{\rho_i})(s_i + \sigma'_{\lambda_i} + \sigma'_{\rho_i}) - (h_i + \sigma_{\lambda_i} + \sigma_{\rho_i})(t_i + \sigma'_{\lambda_i} - \sigma'_{\rho_i}) \Bigg]. \end{aligned} \tag{3.22}$$

Now we define $\sigma_i^{\pm} = \sigma_{\lambda_i} \pm \sigma_{\rho_i}$. After summing over s_i and t_i, the contribution from the scalar fields P to the partition function (3.22) is given by

$$
\begin{aligned}
Z_b^{\mathrm{instP}} = & \prod_i \sum_{\substack{\phi,\chi,\sigma_{\lambda_i},\sigma_{\rho_i},\sigma'_{\lambda_i},\sigma'_{\rho_i} \\ h_i,l_i,u_i}} \frac{1}{8}\sqrt{\frac{4\tau_2^i}{\xi_i}} \exp\Bigg[-\frac{\pi\xi_i}{4}\Bigg((h_i+\sigma_i^+)^2\frac{|\tau^i|^2}{\tau_2^i} \\
& -2\sigma_i'^+(h_i+\sigma_i^+)\frac{\tau_1^i}{\tau_2^i}+(\sigma_i'^+)^2\frac{1}{\tau_2^i}\Bigg)+2\pi i\phi h_i+\frac{i\pi\xi_i}{2}(l_i+\sigma_i^-)\sigma_i'^+ \\
& -\frac{4\pi\tau_2^i}{\xi_i}\Bigg(u_i-\chi+i\xi_i\frac{(h_i+\sigma_i^+)}{4}\frac{\tau_1^i}{\tau_2^i}-\frac{i\xi_i\sigma_i'^+}{4\tau_2^i}-\frac{\xi_i}{4}(l_i+\sigma_i^-)^2\Bigg)\Bigg].
\end{aligned}
\tag{3.23}
$$

On rearranging the sum takes the form

$$
\begin{aligned}
Z_b^{\mathrm{instP}} = & \prod_i \sum_{\substack{\phi,\chi,\sigma_{\lambda_i},\sigma_{\rho_i},\sigma'_{\lambda_i},\sigma'_{\rho_i} \\ h_i,l_i,u_i}} \sqrt{\frac{\tau_2^i}{\xi_i}} \exp\Bigg[-\frac{\pi\xi_i}{4}(h_i+\sigma_i^+)^2 \\
& -4\xi_i\Bigg(\frac{u_i-\chi}{\xi_i}-\frac{1}{4}(l_i+\sigma_i^-)\Bigg)^2+\frac{i\pi\xi_i}{2}(l_i+\sigma_i^-)\sigma_i'^+ \\
& -2\pi\tau_1^i(h_i+\sigma_i^+)\Bigg(u_i-\chi-\frac{\xi_i}{4}(l_i+\sigma_i^-)\Bigg)+2\pi i\phi h_i+2\pi i(u_i-\chi)\sigma_i'^+\Bigg].
\end{aligned}
\tag{3.24}
$$

Similar expressions can be written for the fields Q with h_i replaced by l_i. Combining the contributions from the scalar fields P's and Q's, the instanton part of the partition function for the bosons is written as

$$
\begin{aligned}
Z_b^{\mathrm{inst}} = & \prod_i \sum_{\substack{\phi,\chi,\sigma_{\lambda_i},\sigma_{\rho_i},\sigma'_{\lambda_i},\sigma'_{\rho_i} \\ h_i,l_i,u_i,\widehat{u}_i}} \Bigg(\sqrt{\frac{\tau_2^i}{2\xi_i}} \exp\Bigg[i\pi\xi_i\tau^i\frac{{p_L^i}^2}{2}-i\pi\xi_i\overline{\tau}^i\frac{{p_R^i}^2}{2} \\
& \qquad\qquad +2\pi i(\phi h_i+(u_i-\chi)\sigma_i'^+)\Bigg]\Bigg) \\
& \times\Bigg(\sqrt{\frac{\tau_2^i}{2\xi_i}} \exp\Bigg[i\pi\xi_i\tau^i\frac{{q_L^i}^2}{2}-i\pi\xi_i\overline{\tau}^i\frac{{q_R^i}^2}{2}+2\pi i(\phi l_i+(\widehat{u}_i+\chi)\sigma_i'^+)\Bigg]\Bigg),
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
p_L^i &= \frac{1}{2}(h_i+\sigma_i^+)-2\Bigg(\frac{u_i-\chi}{\xi_i}-\frac{1}{4}(l_i+\sigma_i^-)\Bigg), \\
p_R^i &= \frac{1}{2}(h_i+\sigma_i^+)+2\Bigg(\frac{u_i-\chi}{\xi_i}-\frac{1}{4}(l_i+\sigma_i^-)\Bigg), \\
q_L^i &= \frac{1}{2}(l_i+\sigma_i^-)-2\Bigg(\frac{\widehat{u}_i+\chi}{\xi_i}-\frac{1}{4}(h_i+\sigma_i^+)\Bigg), \\
q_R^i &= \frac{1}{2}(l_i+\sigma_i^-)+2\Bigg(\frac{\widehat{u}_i+\chi}{\xi_i}-\frac{1}{4}(h_i+\sigma_i^+)\Bigg).
\end{aligned}
\tag{3.26}
$$

The sums over h_i, l_i, σ_i^+, σ_i^-, ϕ etc can be replaced by sums over n_i and $\widehat{n}_i$ using the expressions (3.19) and making use of the identity

$$
\sum_{\xi=0}\Bigg(\exp\Bigg(\frac{2\pi i\xi}{n}\Bigg)\Bigg)^j=\sum_{\sigma_n}\exp(2\pi i\sigma_n j)=\begin{cases} n, \text{if } j=0 \mod n \\ 0, \text{ otherwise.}\end{cases}
\tag{3.27}
$$

Consequently, we have

$$\begin{aligned} u_i + \widehat{u}_i &= 0 \quad \mathrm{mod}\ \lambda_i, \\ u_i - \widehat{u}_i - 2\chi &= 0 \quad \mathrm{mod}\ \rho_i, \end{aligned} \tag{3.28}$$

and these criteria are satisfied by the choice

$$\frac{u_i - \chi}{\xi_i} = \frac{1}{2}\left(\frac{\omega_i}{\rho_i} + \frac{\widehat{\omega}_i}{\lambda_i}\right), \quad \frac{\widehat{u}_i + \chi}{\xi_i} = \frac{1}{2}\left(\frac{\omega_i}{\rho_i} - \frac{\widehat{\omega}_i}{\lambda_i}\right). \tag{3.29}$$

where we have replaced the sums over $u_i, \widehat{u}_i, \chi, \sigma'_{\lambda_i}$ and σ'_{ρ_i} by sums over w_i and $\widehat{w}_i \in \mathbb{Z}$. We can now identify the left-moving and right-moving momenta as

$$\begin{aligned} p_L^i &= \frac{n_i}{\lambda_i} - \left(\frac{\omega_i}{\rho_i} + \frac{\widehat{\omega}_i}{\lambda_i}\right). \\ p_R^i &= \frac{\widehat{n}_i}{\lambda_i} + \left(\frac{\omega_i}{\rho_i} + \frac{\widehat{\omega}_i}{\lambda_i}\right). \\ q_L^i &= -\frac{\widehat{n}_i}{\rho_i} - \left(\frac{\omega_i}{\rho_i} - \frac{\widehat{\omega}_i}{\lambda_i}\right). \\ q_R^i &= \frac{n_i}{\rho_i} + \left(\frac{\omega_i}{\rho_i} - \frac{\widehat{\omega}_i}{\lambda_i}\right). \end{aligned} \tag{3.30}$$

The doubled partition function for the bosons can now be written as

$$\begin{aligned} Z_b^{\text{doubled}} =& \prod_i \sum_{p_L^i, p_R^i} \sqrt{2\tau_2^i} \exp\left[i\pi\xi_i\tau^i \frac{{p_L^i}^2}{4} - i\pi\xi_i\overline{\tau}^i \frac{{p_R^i}^2}{4}\right] \\ & \times \sum_{q_L^i, q_R^i} \sqrt{2\tau_2^i} \exp\left[i\pi\xi_i\tau^i \frac{{q_L^i}^2}{4} - i\pi\xi_i\overline{\tau}^i \frac{{q_R^i}^2}{4}\right]. \end{aligned} \tag{3.31}$$

Since the momentum sums are decoupled we can keep only the holomorphic part of the doubled partition function from both Ps and Qs. This gives us the contribution of the bosonic fields to the partition functions on a two-torus.

$$Z_b^{\text{holo}} = \prod_i \sum_{p_L^i, q_R^i} \sqrt{2\tau_2^i} \exp\left[i\pi\xi_i\tau^i \frac{{p_L^i}^2}{4} - i\pi\xi_i\tau^i \frac{{q_R^i}^2}{4}\right]. \tag{3.32}$$

The holomorphic factorization of the "instanton" sum guarantees the inclusion of all spin structures necessary for the chiral bosons. Thus the contributions from the bosonic instanton sector to the partition function give us the sum over the entire momentum lattice.

4. The oscillator contributions

The contribution to the partition function from the oscillator sectors of bosons and fermions can be obtained by evaluating the path integral with the action (2.9). For the bosonic case

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this yields the (squared) partition function of the bosonic string theory [26] on a two-torus. Let us discuss this in brief. The path integral for bosons is

$$Z_b^{\rm osc} = \int DX^1 DX^2 \exp\left[-\frac{\pi}{2}\int d^2z[R_1^2\partial_{\overline{z}}X^1\partial_z X^1 + R_2^2\partial_{\overline{z}}X^2\partial_z X^2]\right], \tag{4.1}$$

which evaluates to

$$Z_b^{\rm osc} = \frac{R_1}{\sqrt{2\det\Box}}\frac{R_2}{\sqrt{2\det\Box}}, \tag{4.2}$$

with $\det\Box = \tau_2^1\eta^2(\tau)\eta^2(\overline{\tau})$, where $\eta(\tau)$ is the Dedekind η-function

$$\eta(\tau) = e^{i\pi/12}\prod_{n>1}(1-e^{2\pi i n\tau}). \tag{4.3}$$

where τ is the modular parameter on the worldsheet torus.

The path integral over the dual fields $\widehat{\phi}$ gives a similar expression, the only difference being that the radii R_1 and R_2 appear in the denominator. Taking all the bosonic contributions into account, the square-root of the path integral contributes the following piece to the partition function

$$Z_b = \prod_i \sum_{p_L^i, q_R^i}\frac{1}{|\eta|^4}\exp\left[i\pi\tau\frac{\left(p_L^i\right)^2}{4} - i\pi\overline{\tau}\frac{\left(q_R^i\right)^2}{4}\right]. \tag{4.4}$$

Since the radii of the two-torus appearing in the expression (4.2) cancel with those appearing in the expression for the dual bosons the final expression above for the bosons is T-duality invariant. Let us now discuss the fermionic contributions. The path integral for the fermions is

$$\begin{aligned} Z_f^{\rm osc} = \int D\psi^1 D\psi^2 D\widetilde{\psi}^1 D\widetilde{\psi}^2 \exp\Big(-\frac{\pi}{2}\int d^2z\big(R_1^2(\psi^1\partial_{\overline{z}}\psi_1 + \widetilde{\psi}^1\partial_z\widetilde{\psi}_1) \\ + R_2^2(\psi^2\partial_{\overline{z}}\psi_2 + \widetilde{\psi}^2\partial_z\widetilde{\psi}_2)\big)\Big) \\ \times \exp\left[\text{topological terms for fermions}\right]. \end{aligned} \tag{4.5}$$

The topological terms for the fermions are total derivatives which do not contribute to the fermionic equations of motion, nor do they contribute to the path integral. In the path integral (4.5) since the two fields ψ and $\widetilde{\psi}$ are decoupled, the partition function is the product of the Pfaffians of the differential operators ∂_z and $\partial_{\overline{z}}$ [30],

$$Z = Pf(\partial_z)Pf(\partial_{\overline{z}}). \tag{4.6}$$

As the product $\partial_z\partial_{\overline{z}}$ is the two-dimensional Laplacian, we get

$$Z = (\det\nabla^2)^{\frac{1}{2}}. \tag{4.7}$$

Now we impose periodicity conditions on the fermions. When translated by a period the fermions pick up a phase.

$$\psi(z+\phi) = e^{2i\pi\nu}\psi(z). \tag{4.8}$$

For $\nu \in \mathbb{Z}$, ψ and $\widetilde{\psi}$ are periodic and for $\nu \in (\mathbb{Z}+1/2)$, both are antiperiodic. The combinations of periodic(P) and antiperiodic(A) boundary conditions for the holomorphic and antiholomorphic fields are used to define the spin structure of the fermions, that is, the Ramond (periodic) and Neveu-Schwarz (antiperiodic) sectors. Again, due to the factorization of the holomorphic and the antiholomorphic parts it suffices to compute the integral for the holomorphic fields only and the partition function is evaluated as

$$Z = |\det \partial_z|^2. \tag{4.9}$$

Evaluating the regularized products with P and A boundary conditions, we obtain

$$\begin{aligned}
(\det \partial_z)_{A,A} &= \frac{\vartheta_3(\tau)}{\eta(\tau)}, \qquad \mathrm{NS-NS}\\
(\det \partial_z)_{A,P} &= \frac{\vartheta_4(\tau)}{\eta(\tau)}, \qquad \mathrm{NS-R}\\
(\det \partial_z)_{P,A} &= \frac{\vartheta_2(\tau)}{\eta(\tau)}, \qquad \mathrm{R-NS}\\
(\det \partial_z)_{P,P} &= \frac{\vartheta_1(\tau)}{\eta(\tau)}. \qquad \mathrm{R-R}
\end{aligned} \tag{4.10}$$

The contributions from the dual fermions are obtained in the same way with the same results. The partition function for the fermions on T^2 is given by the combinations of theta functions. In order to obtain the partition function of type-II theories, however, one now has to impose the GSO projections, as usual.

Upon choosing the GSO projection $\exp(i\pi F) = 1$, we obtain the partition function of the type-IIB theory, *viz.*

$$Z_f^{\rm osc} = \frac{1}{|\eta|^2}|\vartheta_3 - \vartheta_4 - \vartheta_2 + \vartheta_1|^2. \tag{4.11}$$

Choosing, on the other hand, the GSO projection $= \exp(i\pi\widetilde{F}) = (-1)^\alpha$, with $\alpha = 1$ in the R-sector and $\alpha = 0$ in the NS-sector, gives the partition function of the type-IIA theory, *viz.*

$$Z_f^{\rm osc} = \frac{1}{|\eta|^2}(\vartheta_3 - \vartheta_4 - \vartheta_2 + \vartheta_1)(\overline{\vartheta_3} - \overline{\vartheta_4} - \overline{\vartheta_2} - \overline{\vartheta_1}). \tag{4.12}$$

Finally, combining the expressions obtained above, the total partition function for the type-IIB theory is obtained as

$$Z = \prod_i \sum_{n_1,n_2,\omega_1,\omega_2} \frac{1}{|\eta|^6} \exp\left[i\pi\tau\frac{(p_L^i)^2}{2} - i\pi\overline{\tau}\frac{(q_R^i)^2}{2}\right] \times |\vartheta_3 - \vartheta_4 - \vartheta_2 + \vartheta_1|^2. \tag{4.13}$$

The total partition function for the type-IIA theory is obtained similarly, with the afore mentioned GSO projection as

$$Z = \prod_i \sum_{n_1,n_2,\omega_1,\omega_2} \frac{1}{|\eta|^6} \exp\left[i\pi\tau\frac{(p_L^i)^2}{2} - i\pi\overline{\tau}\frac{(q_R^i)^2}{2}\right] \times (\vartheta_3 - \vartheta_4 - \vartheta_2 + \vartheta_1)(\overline{\vartheta_3}-\overline{\vartheta_4} - \overline{\vartheta_2} - \overline{\vartheta_1}). \tag{4.14}$$

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5. Conclusion

To summarize, we have studied the supersymmetric extension of the doubled formalism. We evaluated the partition function of the $\mathcal{N} = (1,1)$ NLSM on a doubled-torus, $T^2 \times \widehat{T}^2$. The superstring partition functions turn out to be the squareroot of the partition function of this theory and are T-duality invariant. For the bosonic case, a topological term had to be added to the NLSM on the doubled torus for quantum consistency. In order to maintain supersymmetry in our case, a supersymmetric extension of the same by fermions is required. However, these extra fermions conspired not to contribute to the partition functions. This supersymmetric model, thus, reproduces the one-loop partition functions of type-II theories exactly. Thus the calculations presented here provide a non-trivial verification of the doubled formalism, extending to the supersymmetric cases. This matching, however, requires a separate choice of the GSO projections as in the traditional formulations of type-II theories. It would be interesting to study the Hilbert space of the NLSM on the doubled torus and compare with the spectra of the type-II theories directly. The fermions in the supersymmetrically extended topological terms may have an interesting role to play in such a study. Extension of this formalism to study superstrings on more complicated spaces, such as K3 and Calabi-Yau manifolds will also be interesting. These will be useful in formulating a T-duality invariant string theory. Finally, formulation presented here can be applied to the case of $\mathcal{N} = (1,0)$ superspace, relevant for the Heterotic strings.

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